

Geometry of the SU(2) Di-Meron Solution

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Z. Naturforsch. **41a**, 571–584 (1986); received December 2, 1985

The geometric properties of the di-meron solution to the SU(2) Yang-Mills equations are studied in detail. The essential geometric structure of this solution is that of a locally symmetric space endowed with a Riemannian structure which is conformally flat. The di-meron solution is representable by an integrable 3-distribution over Euclidean 4-space. The corresponding integral surfaces are obtained in analytic form.

I. Introduction

The relative success of QCD in explaining the hadronic bound states is based mainly upon semi-classical methods or perturbative results. In view of the absence of a rigorous proof for quark confinement, the semi-classical methods have been used in order to get a preliminary qualitative understanding of this fundamental hadronic phenomenon. Since this approximative approach is built upon certain classical field configurations, the interest in finding such classical solutions was considerable in the early phase of the evolution of the non-abelian gauge theories. In the meantime, the relevant field configurations such as instantons and merons, have been discovered and studied in great detail. The application of those solutions to the physics of hadrons has confirmed the original expectations even at the early stage [1] when it was recognized that the instanton gas is responsible for the complex structure of the vacuum (e.g. θ -vacua). Further, the dissociation of the instantons into merons ("half-instantons") was described in terms of a plasmalike phase confining the quarks.

Besides this physical background, the classical solution have been found to be interesting from the purely mathematical point of view, too [2]. Especially the conformal properties of instantons [3, 4] and merons [5] have been studied in some detail, which in turn led to a better understanding and handling of those field configurations in physics.

The present paper intends to demonstrate the importance of the conformal properties of the di-meron solution. In the course of the investigations

an intrinsic Riemannian structure of the conformally flat type arises in a most natural way. This conformal structure is closely related to the extrinsic geometric properties of the di-meron configuration. It turns out that this field is geometrically representable by an integrable, 3-dimensional distribution \bar{A} over Euclidean 4-space E_4 . In this way, the present considerations represent a further example for the relevance of geometric methods for obtaining and studying the solutions to non-linear field equations (cf. [6–10]).

In more detail, our results are as follows:

In Sect. II, the notion of a "trivializable" solution $A(x)$ to the free Yang-Mills equations $D \cdot F(x) = 0$ is introduced. By the very definition of trivializability, a (gauge) tensor object $B(x)$ may be associated to any such a trivializable field configuration $A(x)$ so that both objects coincide ($A(x) \equiv B(x)$) when a certain gauge is applied ("positive gauge"). The field strengths $F(x)$ turn out as a quadratic form of the B -fields, which themselves have vanishing alternating derivatives $D \wedge B(x) = 0$.

It is further shown that a trivializable field configuration, given in the positive gauge, has always half the value of a pure gauge: $\mathcal{A}_\mu = \frac{1}{2} X^{-1} \partial_\mu X$; here X is an element of a four-dimensional, real representation of the gauge group SU(2). Thus, the famous factor $\frac{1}{2}$ turns out as the kinematical characteristic of trivializability. The local isomorphism $SO(4) \sim SU(2) \times SU(2)$ plays an important part for this mechanism. The latter fact also implies that any trivializable A may be represented geometrically by some 3-distribution \bar{A} over E_4 . If \bar{A} is integrable, the configuration $A(x)$ is even represented by a system of 3-surfaces, and the tensor objects $B(x)$ acquire the meaning of extrinsic curvature of the integral surfaces.

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In Sect. III, the specific dependence of the field strengths F upon the extrinsic curvature field B is used in order to transform the Yang-Mills equations into the corresponding condition of the B -fields. The latter condition suggests, in a most natural way, the introduction of a Riemannian structure over E_4 such that the extrinsic curvature B becomes constant in the generally covariant sense (gauge plus coordinate covariance). The conformal structure now enters the considerations by the observation that the Yang-Mills equations are automatically satisfied by choosing a Riemannian metric G which is conformally flat. A (“generally relativistic”) $GL(4, \mathbb{R})$ transformation casts the Riemannian connection Γ into the $\mathfrak{SD}(4)$ valued version ω . The supposition is proposed that the trivializable solution $A(x)$ constitutes the $\mathfrak{SU}(2)$ subgeometry of the Riemannian $SO(4)$ structure, i.e. $A = \omega|_{\mathfrak{SU}(2)}$.

In Sect. IV, we verify this supposition by looking for the most general (topologically non-trivial [11]) solution due to the conformally flat ansatz. It turns out that the conformal geometry is determined by a covariantly constant vector field $p(x)$ (“characteristic vector”) which fixes a further 3-distribution $\bar{\mathcal{A}}$. Both the projector (\bar{P}) onto $\bar{\mathcal{A}}$ as well as the Riemannian curvature tensor (R) are composed quadratically by the extrinsic curvature fields B . Since the Yang-Mills fields F exhibit the same dependence upon the B -fields, the $\mathfrak{SU}(2)$ part of the Riemannian R is just given by the Yang-Mills fields F . This enables one to express the extrinsic curvature B in terms of the characteristic vector p such that the pair of fields $B = B(p)$ and $A = \omega|_{\mathfrak{SD}(3)}$ describes the most general trivializable solution of the conformal type. This solution turns out as the well-known di-meron configuration consisting essentially of two hedgehog fields, and the conformal factor is the (Euclidean) square length of the characteristic vector p .

In Sect. V, we turn to the geometric representation of the conformal solutions. A closed first-order equation for the normal $\hat{n}(x)$ to the distribution $\bar{\mathcal{A}}$ is obtained, and $\hat{n}(x)$ is found explicitly by means of a quaternion formalism. The integral surfaces of $\bar{\mathcal{A}}$ for the di-meron solution are obtained in analytic form. Thus, the melting of the single meron surfaces into the di-meron combination becomes visible in detail. The integral lines of the characteristic vector p are geodesic lines with respect to the conformally flat Riemannian connection Γ and determine the

(Euclidean) parallel transport of the normal \hat{n} to $\bar{\mathcal{A}}$. These characteristic lines may be considered as foliations of three-dimensional closed surfaces suspending the merons. The meron “bag” of Glimm and Jaffe [12] turns out as a special case contained in the more general set of 3-surfaces built by the characteristic lines.

In Sect. VI, the paper is closed by demonstrating that the generic multiple meron solution (more than two merons on a straight line) is also of the trivializable type. In comparison to the di-meron case, one is only forced to give up a certain condition of covariant constancy.

It seems unclear at present, whether the most general situation, where the merons are dislocated arbitrarily over E_4 represents a trivializable configuration of the conformal type, too.

II. Trivializable Configurations

In the following we are concerned exclusively with “trivializable” solutions $A_{i\mu}$ of the free Yang-Mills equations

$$D_\mu F_i^{\mu\nu} = 0. \quad (\text{II.1})$$

Here, the field strengths F are constructed by means of the gauge potentials A as usual

$$F_{i\mu\nu} = \partial_\mu A_{i\nu} - \partial_\nu A_{i\mu} + \varepsilon_i^{jk} A_{j\mu} A_{k\nu}. \quad (\text{II.2})$$

A $\mathfrak{SU}(2)$ gauge field A is called *trivializable*, if these “potentials” $A_{i\mu}$ form the connection coefficients of the $\mathfrak{SD}(3)$ restriction of some trivial $\mathfrak{SD}(4)$ connection $\hat{\omega}_\mu$, symbolically: $A_{i\mu} = \hat{\omega}_\mu|_{\mathfrak{SD}(3)}$.

In this case, there exist [13] objects $B_{i\mu}$ transforming tensorially with respect to a change of gauge

$$B'_{i\mu} = B_{j\mu} \bar{S}^j_i, \quad (\text{II.3})$$

whereas the gauge potentials $A_{i\mu}$ behave like the corresponding connection fields

$$A'_{i\mu} = A_{j\mu} \bar{S}^j_i - \frac{1}{2} \varepsilon^{jki} (\bar{S}^{-1})^k_l \partial_\mu \bar{S}^l_j. \quad (\text{II.4})$$

The $SO(3)$ element $\bar{S} = \{\bar{S}^i_j\}$ is the adjoint representation of the corresponding $SU(2)$ element denoted hereafter by X . As a consequence, the embedding $\mathfrak{SD}(4)$ connection $\hat{\omega}_\mu$ may be written as

$$\hat{\omega}_\mu = A_{i\mu} L^i + B_{i\mu} l^i, \quad (\text{II.5})$$

where the $\mathfrak{SD}(4)$ generators L^i ($i = 1, 2, 3$) span the standard $\mathfrak{SD}(3)$ subalgebra [14] of $\mathfrak{SD}(4)$:

$$[L^i, L^j] = \varepsilon^{ij}_k L^k. \quad (\text{II.6})$$

Further, the $\{l^i\}$ are the “boost operators” of the four-dimensional rotation group SO(4):

$$[l^i, l^j] = \varepsilon^{ij}_k l^k, \quad (\text{II.7})$$

$$[l^i, L^j] = \varepsilon^{ij}_k l^k. \quad (\text{II.8})$$

The trivializable gauge potentials as defined above exhibit some specific properties which shall be enumerated briefly.

First, the triviality of the embedding connection $\hat{\omega}_\mu$ implies that the curvature $\hat{\Omega}_{\mu\nu}$ of $\hat{\omega}$ vanishes:

$$\hat{\Omega}_{\mu\nu} \equiv \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu + [\hat{\omega}_\mu, \hat{\omega}_\nu] = 0. \quad (\text{II.9})$$

Since the $\mathfrak{SO}(3)$ part of $\hat{\Omega}$ must vanish separately, one is led immediately to the following special shape of the (trivializable) field strengths F (II.2):

$$F_{i\mu\nu} = -\varepsilon_i{}^{jk} B_{j\mu} B_{k\nu}. \quad (\text{II.10})$$

Further, the vanishing of the boost part yields

$$D_\mu B_{i\nu} = D_\nu B_{i\mu}. \quad (\text{II.11})$$

On account of the latter relations [15] the Bianchi identity

$$D_\mu * F_i{}^{\mu\nu} = 0 \quad (\text{II.12})$$

becomes immediately obvious.

Further properties of trivializable potentials $A_{i\mu}$ arise from the fact that the vanishing of curvature $\hat{\Omega}$ restricts $\hat{\omega}$ to a pure gauge:

$$\hat{\omega}_\mu = A^{-1} \cdot \partial_\mu A, \quad A \in \text{SO}(4). \quad (\text{II.13})$$

Thus, any choice of the SO(4) element A fixes the $\mathfrak{SU}(2)$ gauge for the potentials $A_{i\mu}$.

In the following, we shall need three special gauges. The first two gauges are obtained by observing that SO(4) is locally isomorphic to the product manifold $\text{SU}(2) \times \text{SU}(2)$. Let the corresponding algebras $\mathfrak{SU}^\pm(2)$ be spanned by the generators X^i, Y^i , resp.,

$$X^i = \frac{1}{2}(L^i + l^i) \in \mathfrak{SU}^+(2), \quad (\text{a})$$

$$Y^i = \frac{1}{2}(L^i - l^i) \in \mathfrak{SU}^-(2), \quad (\text{b}) \quad (\text{II.14})$$

such that $\mathfrak{SO}(4) = \mathfrak{SU}^+(2) \oplus \mathfrak{SU}^-(2)$ and

$$[X^i, Y^j] = 0, \quad (\text{a})$$

$$[X^i, X^j] = \varepsilon^{ij}_k X^k, \quad (\text{b})$$

$$[Y^i, Y^j] = \varepsilon^{ij}_k Y^k. \quad (\text{c}) \quad (\text{II.15})$$

The “positive gauge” (${}^+A_{i\mu}$) is now determined by choosing the SO(4) element A (II.13) to be con-

tained in the $\text{SU}^+(2)$ factor of SO(4):

$$A \rightarrow X = \exp[-v_i X^i] \in \text{SU}^+(2). \quad (\text{II.16})$$

This choice yields for the trivial $\hat{\omega}_\mu$ (II.13)

$$\hat{\omega}_\mu = {}^+A_{i\mu} X^i, \quad (\text{II.17})$$

where the (trivial) $\mathfrak{SU}(2)$ potential ${}^+A_{i\mu}$ is given by

$${}^+A_{i\mu} = 4(Y_i)_{\sigma\tau} \hat{n}^\sigma \partial_\mu \hat{n}^\tau \quad ({}^+\hat{F}_{i\mu\nu} = 0). \quad (\text{II.18})$$

Here, we have introduced the four-dimensional unit vector \hat{n}^σ ($\hat{n}^\sigma \hat{n}_\sigma = -1$) in favor of the SU(2) group parameters $\{v^i\}$ (cf. (II.16)):

$$\hat{n}^\sigma = \{\cos v/2, \bar{v}^i \sin v/2\}, \quad v^i v_i = -v^2 (< 0), \quad \bar{v}^i = v^i/v. \quad (\text{II.19})$$

Considering now the $\mathfrak{SO}(3)$ and boost parts of $\hat{\omega}_\mu$ (II.17) each separately, we readily find by means of the decompositions (II.14)

$${}^+A_{i\mu} = {}^+B_{i\mu} = \frac{1}{2} {}^+A_{i\mu}. \quad (\text{II.20})$$

So we see that in the positive gauge, the tensor objects $B_{i\mu}$ just agree with the gauge potentials $A_{i\mu}$ and both take half the value of a trivial potential ($A_{i\mu}$). This may be cast into a more concise form by defining the $\mathfrak{SU}^+(2)$ valued gauge potential \mathcal{A}_μ according to

$$\mathcal{A}_\mu = A_{i\mu} X^i \quad (\text{II.21})$$

and then simply write the positive gauge ${}^+\mathcal{A}_\mu$ as

$${}^+\mathcal{A}_\mu = \frac{1}{2} X^{-1} \partial_\mu X. \quad (\text{II.22})$$

This shape of the gauge potential was used in the very beginning of the meron research; the origin of the factor $\frac{1}{2}$ has now been traced back to the above trivializability property! Moreover, it is seen from (II.22) that the general trivializable potential \mathcal{A}_μ in the positive gauge is always the $\mathfrak{SU}^+(2)$ restriction of a non-trivial $\mathfrak{SO}(4)$ connection which itself is half the value of a pure SO(4) gauge, i.e.

$${}^+\mathcal{A}_\mu = \frac{1}{2} (A^{-1} \partial_\mu A)|_{\mathfrak{SU}^+(2)} \quad (A \in \text{SO}(4)). \quad (\text{II.23})$$

This follows readily from the fact that any SO(4) element A may be locally decomposed into its $\text{SU}^+(2)$ factors according to

$$A = X \cdot Y = Y \cdot X, \quad X \in \text{SU}^+(2), \quad Y \in \text{SU}^-(2). \quad (\text{II.24})$$

The “negative gauge” (${}^-A_{i\mu}$) is the $\mathfrak{SU}^-(2)$ analogue of the positive gauge mentioned so far. In

this case one replaces the $SU^+(2)$ element X (II.16) by Y

$$Y = \exp[v_i Y^i] \in SU^-(2) \quad (\text{II.25})$$

and then finds analogously to (II.22)

$${}^{\sim}\mathcal{A}_\mu = {}^{\sim}A_{i\mu} X^i = \frac{1}{2} {}^{\sim}X \partial_\mu X^{-1}. \quad (\text{II.26})$$

Here, we find that the gauge potentials ${}^{\sim}A_{i\mu}$ differ from the tensor objects ${}^{\sim}B_{i\mu}$ only in sign:

$$\begin{aligned} {}^{\sim}A_{i\mu} &= -{}^{\sim}B_{i\mu} = \frac{1}{2} {}^{\circ}\tilde{A}_{i\mu}, \quad (\text{a}) \\ {}^{\circ}\tilde{A}_{i\mu} &= 4(X_i)_{\varrho\sigma} \hat{n}^\varrho \partial_\mu \hat{n}^\sigma. \quad (\text{b}) \end{aligned} \quad (\text{II.27})$$

Finally, let us turn to the third gauge which is relevant for the present considerations: The “neutral gauge” (${}^nA_{i\mu}$). This gauge [16] is a kind of mean value of the first two gauges and is obtained by choosing the $SO(4)$ element A (II.13) as

$$A \rightarrow Z = \exp\left[\frac{1}{2} v_i L^i\right] = (X \cdot Y)^{-\frac{1}{2}}. \quad (\text{II.28})$$

This yields for the gauge potentials

$${}^nA_{i\mu} = \left(1 - \cos \frac{v}{2}\right) \varepsilon_i{}^{jk} \bar{v}_k \partial_\mu \bar{v}_j, \quad (\text{II.29})$$

and for the tensor objects:

$${}^nB_{i\mu} = \bar{v}_i \partial_\mu \frac{v}{2} + \tan \frac{v}{2} D_\mu \bar{v}_i. \quad (\text{II.30})$$

The latter relation shows that $\mathbf{v} = \{v^i\}$ is an $SO(3)$ gauge scalar field.

Clearly, these three gauges are linked by the appropriate $SU(2)$ gauge transformations, i.e.

$${}^{\sim}\mathcal{A}_\mu = X \cdot {}^+\mathcal{A}_\mu \cdot X^{-1} + X \cdot \partial_\mu X^{-1}, \quad (\text{II.31})$$

whereas the neutral gauge ${}^n\mathcal{A}_\mu = {}^nA_{i\mu} X^i$ is related to the positive gauge ${}^+\mathcal{A}_\mu$ through

$${}^n\mathcal{A}_\mu = X^{\frac{1}{2}} \cdot {}^+\mathcal{A}_\mu \cdot X^{-\frac{1}{2}} + X^{\frac{1}{2}} \cdot \partial_\mu X^{-\frac{1}{2}}. \quad (\text{II.32})$$

The gauge equivalence of the three potentials $\{{}^+\mathcal{A}_\mu, {}^{\sim}\mathcal{A}_\mu, {}^n\mathcal{A}_\mu\}$ may be expressed also in geometric terms. Indeed, the triviality of the $\mathfrak{SD}(4)$ connection $\hat{\omega}_\mu$ (II.5) implies that there exists an orthonormal tetrad $\mathbf{e} = \{e_\mu, (e_\mu \cdot e_\nu) = g_{\mu\nu}\}$ moving over Euclidean four-space E_4 such that

$$\partial_\mu e_\lambda = e_\nu \hat{\omega}_{\lambda\mu}^\nu. \quad (\text{II.33})$$

By virtue of this geometric interpretation of the embedding $\mathfrak{SD}(4)$ connection $\hat{\omega}_\mu$, the gauge potentials $A_{i\mu}$ and tensor objects $B_{i\mu}$ are representable

now in terms of the tetrad fields $e_\mu(x)$ as

$$A_{i\mu} = \frac{1}{2} \varepsilon_i{}^{jk} e_j \cdot \partial_\mu e_k, \quad (\text{a})$$

$$B_{i\mu} = \hat{n} \cdot \partial_\mu e_i. \quad (\text{b}) \quad (\text{II.34})$$

Here, we have identified $\mathbf{e}_0 \equiv \hat{n}$.

Thus, the geometric meaning of the trivializable $A_{i\mu}$ becomes manifest now: The gauge potentials $A_{i\mu}$ describe the parallel transport of an orthogonal triad $\bar{\mathbf{e}} = \{e_i, i = 1, 2, 3\}$ in a 3-dimensional distribution $\bar{\Delta}$ over E_4 whose unit normal is just given by \hat{n} ($\equiv \mathbf{e}_0$). The tensor objects $B_{i\mu}$ acquire the meaning of extrinsic curvature fields for this distribution $\bar{\Delta}$. An $SO(3)$ rotation of the triad $\bar{\mathbf{e}}$,

$$e'_i = \bar{S}^j_i e_j, \quad (\text{II.35})$$

immediately leads to the transformation laws (II.3, 4). Clearly, the generating $SO(4)$ elements X, Y, Z for the three different gauges, mentioned above, just differ by the corresponding $SO(3)$ element $S = \exp[v_i L^i] \in SO(4)$:

$$Y = X \cdot S, \quad (\text{a})$$

$$Z = X^{-1} \cdot S^{-\frac{1}{2}}. \quad (\text{b}) \quad (\text{II.36})$$

If, additionally, the distribution $\bar{\Delta}$ is integrable, the trivializable solutions $A_{i\mu}$ of the Yang-Mills equations are representable by a system of 3-surfaces with unit normal \hat{n} and tangent vectors \mathbf{e}_i ($\hat{n} \cdot \mathbf{e}_i = 0$). We are returning to this point below when looking for the di-meron surfaces.

III. The Conformally Flat Geometry

On the basis of the preceeding considerations we can now demonstrate how an intrinsic conformal structure arises very naturally as consequence of the trivializability property.

By virtue of the special shape (II.10) of the field strengths F for a trivializable solution, the Yang-Mills equations (II.1) read

$$\varepsilon_i{}^{jk} B_k{}^\mu \{D_\mu B_{j\nu} - g_{\mu\nu} (D^\lambda B_{j\lambda})\} = 0. \quad (\text{III.1})$$

Therefore, we automatically obtain a solution to the Yang-Mills equations if we succeed to solve the following equation for the extrinsic curvature fields $B_{j\nu}$:

$$D_\mu B_{j\nu} - g_{\mu\nu} (D^\lambda B_{j\lambda}) = B_{j\lambda} T^\lambda_{\nu\mu}, \quad (\text{III.2})$$

where $T_{\lambda\nu\mu} = g_{\lambda\sigma} T^{\sigma}_{\nu\mu}$ is a totally symmetric tensor

$$T_{\lambda\nu\mu} = T_{\lambda\mu\nu} = T_{\mu\lambda\nu}. \quad (\text{III.3})$$

At this stage, $T_{\lambda\nu\mu}$ is still arbitrary but clearly one expects that some integrability condition of (III.2) cuts $T_{\lambda\nu\mu}$ down to a special form.

Contracting (III.2) yields for the covariant derivative of the extrinsic curvature fields

$$D_{\mu} B_{j\nu} - B_{j\lambda} \Gamma^{\lambda}_{\nu\mu} = 0, \quad (\text{a})$$

$$\Gamma^{\lambda}_{\nu\mu} = T^{\lambda}_{\mu\nu} - \frac{1}{3} T^{\lambda\sigma}_{\sigma} g_{\mu\nu}. \quad (\text{b}) \quad (\text{III.4})$$

It is strongly suggestive now to interpret the $\Gamma^{\lambda}_{\nu\mu}$ as the connection coefficients of a Riemannian structure over E_4 . In this sense, the corresponding coordinate covariant derivative acting on some vector field is denoted henceforth as ∇_{μ} :

$$\nabla_{\mu} p_{\nu} \equiv \partial_{\mu} p_{\nu} - \Gamma^{\lambda}_{\nu\mu} p_{\lambda}. \quad (\text{III.5})$$

The coordinate covariant derivative ∇_{μ} may be combined with the gauge covariant derivative D_{μ} of some gauge scalar v (cf. (II.30))

$$D_{\mu} v_i = \partial_{\mu} v_i + \varepsilon_i^{jk} A_{j\mu} v_k \quad (\text{III.6})$$

into a generally covariant derivative \mathcal{D}_{μ} to be defined as follows:

$$\mathcal{D}_{\mu} B_{j\nu} = D_{\mu} B_{j\nu} - B_{j\lambda} \Gamma^{\lambda}_{\nu\mu} \\ = \nabla_{\mu} B_{j\nu} + \varepsilon_j^{kl} A_{k\mu} B_{l\nu}. \quad (\text{III.7})$$

The content of (III.4) is then nothing else than the generally covariant constancy of the extrinsic curvature fields:

$$\mathcal{D}_{\mu} B_{j\nu} = 0. \quad (\text{III.8})$$

Moreover, this condition of generally covariant constancy immediately implies the constancy of the $\mathfrak{SU}(2)$ field strengths F (II.10) in the same general sense

$$\mathcal{D}_{\lambda} F_{i\mu\nu} = 0. \quad (\text{III.9})$$

Obviously, this is an even stronger condition upon the field strengths than the generally covariant Yang-Mills equations would demand [17]:

$$\mathcal{D}_{\mu} F_i^{\mu\nu} = 0 \quad (F_i^{\mu\nu} = G^{\mu\lambda} G^{\nu\sigma} F_{i\lambda\sigma}). \quad (\text{III.10})$$

Here, the indices have been raised by means of the Riemannian metric $G^{\mu\nu}$ which we assume to underly the connection coefficients $\Gamma^{\lambda}_{\nu\mu}$ in the usual way

$$\Gamma^{\lambda}_{\nu\mu} = \frac{1}{2} G^{\lambda\sigma} (G_{\sigma\mu,\nu} + G_{\sigma\nu,\mu} - G_{\nu\mu,\sigma}). \quad (\text{III.11})$$

So we see that a solution of this type satisfies the Yang-Mills equations in both the gauge covariant (II.1) and the generally covariant form (III.10).

In the present paper, we do not deal with the most general case of a Riemannian structure but restrict ourselves to the conformal type of solutions.

This case is obtained if we assume the Riemannian metric G to be conformally equivalent to the Euclidean metric g ,

$$G_{\mu\nu} = \chi^2 g_{\mu\nu}, \quad (\text{III.12})$$

which gives

$$\Gamma^{\lambda}_{\nu\mu} = g^{\lambda}_{\nu} q_{\mu} + g^{\lambda}_{\mu} q_{\nu} - g_{\nu\mu} q^{\lambda}, \quad (\text{a}) \quad (\text{III.13})$$

$$q_{\nu} = \chi_{,\nu} / \chi, \quad (\text{b})$$

and

$$T_{\lambda\nu\mu} = q_{\lambda} g_{\nu\mu} + q_{\mu} g_{\lambda\nu} + q_{\nu} g_{\mu\lambda}. \quad (\text{III.14})$$

Thus, the symmetry condition (III.3) is automatically satisfied for any conformal metric G (III.12). Observe also that in the conformal case the generally covariant Yang-Mills equations (III.10) imply automatically the gauge covariant version (II.1) independently of the trivializable symmetric ansatz (III.2, 3).

There is an interesting problem now which may be settled strictly for the present conformal type of solutions. It consists in the question how the SU(2) structure, based upon the trivializable solutions $A_{i\mu}$ is linked to the (Riemannian) SO(4) structure, which is based upon the connection coefficients $\Gamma^{\lambda}_{\nu\mu}$ emerging in our special ansatz (III.2–4).

Indeed, we shall prove in the following that the trivializable solutions $A_{i\mu}$ corresponding to the conformal ansatz (III.13) just constitute the SU(2) subgeometry of the conformally flat G1(4, \mathbb{R}) structure based on the connection coefficients Γ . This means concretely that the $\mathfrak{SU}(2)$ restriction \mathcal{A}_{μ} of the $\mathfrak{SO}(4)$ valued gauge copy ω_{μ} of the G1(4, \mathbb{R}) matrix $\Gamma_{\mu} = \{\Gamma^{\lambda}_{\nu\mu}\}$ just agrees with the trivializable solutions \mathcal{A}_{μ} for the Yang-Mills equations

$$\mathcal{A}_{\mu} = \omega_{\mu}|_{\mathfrak{SU}(2)}. \quad (\text{III.15})$$

As a first step for this proof, we quite generally perform a transition from the coordinate basis $\{\vec{\partial}_{\mu}\}$ used so far,

$$G(\vec{\partial}_{\mu}, \vec{\partial}_{\nu}) = G_{\mu\nu}, \quad (\text{III.16})$$

to some orthogonal tetrad $\{E_{\alpha}\}$,

$$G(E_{\alpha}, E_{\beta}) = g_{\alpha\beta}. \quad (\text{III.17})$$

Thus, the Riemannian analogue of the parallel transport law over flat space (II.33) reads now for the coordinate basis $\{\vec{\partial}_\mu\}$

$$\nabla_\mu \vec{\partial}_\nu = \vec{\partial}_\lambda \Gamma^\lambda_{\nu\mu}. \quad (\text{III.18})$$

On the other hand, the use of the orthonormal tetrad $\{E_\alpha\}$ yields

$$\nabla_\mu E_\alpha = E_\beta \omega^\beta_{\alpha\mu}. \quad (\text{III.19})$$

If the change of basis $\vec{\partial}_\mu \rightarrow E_\alpha$ is specified by the transition coefficients $E := \{E^\mu_\alpha\}$,

$$E_\alpha = E^\mu_\alpha \vec{\partial}_\mu, \quad (\text{III.20})$$

the $\text{Gl}(4, \mathbb{R})$ matrix $\Gamma_\mu = \{\Gamma^\lambda_{\nu\mu}\}$ (III.18) is transformed into the $\mathfrak{SD}(4)$ element $\omega_\mu = \{\omega^\beta_{\alpha\mu}\}$ (III.19) according to

$$\omega_\mu = E^{-1} \cdot \Gamma_\mu \cdot E + E^{-1} \cdot \partial_\mu E. \quad (\text{III.21})$$

In order to find from this general relation the $\mathfrak{SD}(4)$ connection due to our conformal ansatz (III.13), we have to restrict the $\text{Gl}(4, \mathbb{R})$ matrix E to the conformal subgroup:

$$E = \chi^{-1} A \quad (A \in \text{SO}(4)). \quad (\text{III.22})$$

Clearly, this leads to a conformal metric (III.12) as a consequence of (III.16, 17):

$$G_{\mu\nu} = g_{\alpha\beta} (E^{-1})^\alpha_\mu (E^{-1})^\beta_\nu = \chi^2 g_{\mu\nu}. \quad (\text{III.23})$$

Further, the $\mathfrak{SD}(4)$ representative ω_μ (III.21) of the Riemannian connection matrix Γ_μ becomes in this conformal case (III.22)

$$\omega^\alpha_{\beta\mu} = -q_\sigma (A^{-1})^\alpha_\nu (L^\sigma_\mu)^\nu_\lambda A^\lambda_\beta + (A^{-1})^\alpha_\nu \partial_\mu A^\nu_\beta, \quad (\text{III.24})$$

where the L^α_β denote the standard $\mathfrak{SD}(4)$ generators [14].

The $\mathfrak{SD}(4)$ connection ω_μ (III.24) may be simplified by exploiting the arbitrariness of the $\text{SO}(4)$ element A , which may be always gauged into the identity ($A \rightarrow 1$). This yields

$$\omega^\alpha_{\beta\mu} = -q_\sigma (L^\sigma_\mu)^\alpha_\beta. \quad (\text{III.25})$$

Now we remember our original aim which was to look for the $\mathfrak{SU}(2)$ part (\mathcal{A}_μ) of ω_μ (III.21). Some simple calculations [18] yield for the appropriate decomposition of (III.25)

$$\begin{aligned} \omega_\mu &= {}^x A_{i\mu} X^i + {}^y A_{i\mu} Y^i \equiv \mathcal{A}_\mu + {}^y A_{i\mu} Y^i, \\ {}^x A_{i\mu} &= 2(X_i)_{\sigma\mu} q^\sigma, \quad (\text{a}) \\ {}^y A_{i\mu} &= 2(Y_i)_{\sigma\mu} q^\sigma. \quad (\text{b}) \end{aligned} \quad (\text{III.26})$$

Thus, we have obtained the $\mathfrak{SU}(2)$ part \mathcal{A}_μ (III.15) of the conformal Riemannian connection Γ_μ ; and if our original assertion were true, we must be able to identify these $\mathfrak{SU}(2)$ potentials (${}^x A_{i\mu}$) as a gauge copy of the extrinsic curvature fields $B_{i\mu}$ emerging as solutions of (III.8). We are going to prove this now by explicitly constructing the most general conformal solutions $B_{i\mu}(x)$ to (III.8) and then identifying them as the tensor objects associated to the conformal potentials ${}^x A_{i\mu}$ (III.26a).

IV. The Conformal Solutions

The conformal ansatz (III.13) suggests to look for the most general solution of this type. We are constructing now this solution in four steps. Thereby, we shall recover some of its characteristic geometric properties which, in turn, shall become useful when dealing with its geometric representation in the next section.

1. Characteristic Lines

At the end of Sect. II, we have already mentioned that any trivializable solution may be represented geometrically by some 3-distribution $\tilde{\mathcal{A}}$. Such a $\tilde{\mathcal{A}}$ is characterized uniquely by its unit normal \hat{n} . Further, any unit field $\hat{n}(x)$ is determined by the three $\text{SU}(2)$ parameters $v^i(x)$ (II.19). Hence, $\hat{n}(x)$ is parallel transported (with respect to the canonical connection $\tilde{\omega}_\mu$ (II.33)) along the intersections of the three 3-surfaces $v^i(x) = \text{const}$ ($i = 1, 2, 3$). These intersections constitute a system of “characteristic lines” which we consider as the integral lines of some vector field $p(x)$ (“characteristic vector field”),

$$p^\mu \partial_\mu \hat{n} = 0. \quad (\text{IV.1})$$

Because of the specific dependence of the extrinsic curvature fields $B_{i\mu}$ (II.34b) upon the unit normal \hat{n} we readily deduce from (IV.1)

$$B_{i\mu} p^\mu = 0. \quad (\text{IV.2})$$

This means that the three curvature fields $B_{i\mu}$ ($i = 1, 2, 3$), which we assume to be linearly independent, span the distribution ($\tilde{\mathcal{A}}$, say) which has p as its normal vector. In other words: there is an automorphism $[B]$ of $\tilde{\mathcal{A}}$ which is determined by the

corresponding fields $B_{\mu\nu}(x)$,

$$B_{\mu\nu} := B_{i\mu} B_{\nu}^i, \quad (\text{a})$$

$$B_{\mu\nu} p^\nu = 0, \quad (\text{b})$$

$$\nabla_\sigma B_{\mu\nu} = 0. \quad (\text{c}) \quad (\text{IV.3})$$

The covariant constancy (IV.3c) follows readily from the corresponding property of the individual fields $B_{i\mu}$ (III.8).

Since the (Euclidean) length of the characteristic vector field \mathbf{p} is arbitrary, we normalize it to unity with respect to the Riemannian metric [17] G (III.12):

$$p^\nu p_\nu \equiv G^{\mu\nu} p_\mu p_\nu = \chi^{-2} p^\nu p_\nu = -1. \quad (\text{IV.4})$$

On the other hand, the generally covariant constancy of the extrinsic curvature fields (III.8) may be combined with the orthogonality relation (IV.2) to give (cf. (III.5))

$$\nabla_\nu p^\mu = 0, \quad (\text{IV.5})$$

i.e. the characteristic lines are geodesics with respect to the Riemannian connection Γ_μ (III.13). Further, it follows from the covariant constancy of the characteristic vector \mathbf{p} (IV.5) that the coordinate covariant projector $\hat{P}_{\mu\nu}$,

$$\hat{P}_{\mu\nu} := G_{\mu\nu} + p_\mu p_\nu, \quad (\text{a})$$

$$\hat{P}_{\mu\nu} = \hat{P}_\mu{}^\lambda \hat{P}_{\lambda\nu}, \quad (\text{b}) \quad (\text{IV.6})$$

is also constant:

$$\nabla_\lambda \hat{P}_{\mu\nu} = 0. \quad (\text{IV.6c})$$

2. Riemannian in Terms of Extrinsic Curvature

Next we consider the Riemannian R due to the connection coefficients Γ (III.13):

$$R^\lambda{}_{\nu\sigma\mu} = \partial_\sigma \Gamma^\lambda{}_{\nu\mu} - \partial_\mu \Gamma^\lambda{}_{\nu\sigma} + \Gamma^\lambda{}_{\rho\sigma} \Gamma^\rho{}_{\nu\mu} - \Gamma^\lambda{}_{\rho\mu} \Gamma^\rho{}_{\nu\sigma}. \quad (\text{IV.7})$$

We shall show that this Riemannian is essentially a quadratic form of the extrinsic curvature fields $B_{i\mu}$. Observe that the field strengths are also quadratic with respect to the $B_{i\mu}$. However, the similarities between R and F are even closer: We shall be able to show that the Riemannian R is covariantly constant just as is F (cf. (III.9)).

For this purpose, we remember the fact that the generally covariant Riemannian \mathcal{R} emerges as the

result of the alternating double derivative

$$(\mathcal{D}_\sigma \mathcal{D}_\mu - \mathcal{D}_\mu \mathcal{D}_\sigma) B_{j\nu} = -\mathcal{R}^\alpha{}_{\nu\sigma\mu} B_{j\alpha} \quad (\text{IV.8})$$

which, on account of the constancy of the $B_{i\mu}$ (III.8), immediately leads to

$$B_{j\alpha} \mathcal{R}^\alpha{}_{\nu\sigma\mu} = 0. \quad (\text{IV.9})$$

The Riemannian \mathcal{R} , emerging here, is due to the generally covariant derivative \mathcal{D}_μ (III.7) which itself is a combination of the gauge covariant derivative D_μ (III.6) and its coordinate covariant counterpart ∇_μ (III.5). Therefore, the generally covariant Riemannian \mathcal{R} is also a combination of the two corresponding parts, where the coordinate covariant part (R) is given by (IV.7) whereas the gauge covariant part may be traced back to the identity

$$(D_\sigma D_\mu - D_\mu D_\sigma) B_{j\nu} = \varepsilon_j{}^{kl} F_{k\sigma\mu} B_{lv}. \quad (\text{IV.10})$$

However, the field strengths $F_{k\sigma\mu}$, emerging here, are of the special shape (II.10) as consequence of the trivializability condition. Hence, collecting the gauge and coordinate covariant parts of \mathcal{R} yields then on account of (IV.9) and (IV.2)

$$\begin{aligned} \mathcal{R}^\alpha{}_{\nu\sigma\mu} &= R^\alpha{}_{\nu\sigma\mu} - G^\alpha{}_\mu B_{\sigma\nu} + G^\alpha{}_\sigma B_{\mu\nu} \\ &= p^\alpha H_{\nu\sigma\mu}. \end{aligned} \quad (\text{IV.11})$$

At this stage, the tensor $H_{\nu\sigma\mu}$ is still undetermined. It may be fixed by observing the covariant constancy of the characteristic vector \mathbf{p} (IV.5)

$$(\nabla_\sigma \nabla_\mu - \nabla_\mu \nabla_\sigma) p_\nu = -R^\alpha{}_{\nu\sigma\mu} p_\alpha = 0, \quad (\text{IV.12})$$

which immediately yields for the Riemannian R

$$R^\alpha{}_{\nu\sigma\mu} = B_{\sigma\nu} \hat{P}^\alpha{}_\mu - B_{\mu\nu} \hat{P}^\alpha{}_\sigma. \quad (\text{IV.13})$$

This Riemannian is composed of covariantly constant objects $B_{\mu\nu}$ (IV.3c) and $\hat{P}_{\mu\nu}$ (IV.6c). Therefore it is the curvature tensor of some locally symmetric space [19]:

$$\nabla_\lambda R^\alpha{}_{\nu\sigma\mu} = 0. \quad (\text{IV.14})$$

Clearly, this property strongly reminds us of the analogous feature of the field strengths $F_{i\mu\nu}$ (III.9) and hence suggests the identification of the Yang-Mills fields $F_{i\mu\nu}$ with the $\mathfrak{SU}(2)$ constituents of this Riemannian R . We shall follow this idea subsequently, but first let us draw another conclusion from R (IV.13) concerning the automorphism $[B]$.

Since the Riemannian R (IV.13) is due to a conformally flat metric (III.12), the Weyl conformal

tensor derived from it must vanish. This condition forces $B_{\mu\nu}$ to be proportional to the projector \hat{P} :

$$B_{\mu\nu} = c^{-2} \hat{P}_{\mu\nu}. \quad (\text{IV.15})$$

Further, the factor c must be a constant because both B and \hat{P} are covariantly constant. Hence, the final shape of the Riemannian becomes

$$R^\alpha_{\nu\mu} = c^2 \{B_{\sigma\nu} B^\alpha_{\mu} - B_{\mu\nu} B^\alpha_{\sigma}\}. \quad (\text{IV.16})$$

We are going to look now somewhat closer at the reductive properties of such a Riemannian.

3. $\mathfrak{SU}(2)$ Subgeometry

The preceding results show that the Riemannian R (IV.16) and the field strengths F (II.10) must be intimately related to each other because both are quadratic with respect to the curvature fields $B_{i\mu}$ which themselves are covariantly constant. Therefore, the Yang-Mills field strengths $F_{i\mu\nu}$ must necessarily be identical to the $\mathfrak{SU}(2)$ constituents of the $\mathfrak{SO}(4)$ valued gauge copy Ω of R .

For a proof of this assertion, we apply once more the $\text{G1}(4, \mathbb{R})$ transformation due to the change of basis (III.20) which yields for the curvature R

$$\Omega^\alpha_{\beta\mu\nu} = (E^{-1})^\alpha_{\lambda} R^\lambda_{\sigma\mu\nu} E^\sigma_{\beta}. \quad (\text{IV.17})$$

However, the restriction of the $\text{G1}(4, \mathbb{R})$ transformation E to the conformal shape (III.22) together with the choice of the $\text{SO}(4)$ element A as the identity element makes R identical to Ω . Hence, R may be directly decomposed into its $\mathfrak{SU}(2)$ parts according to

$$R^\alpha_{\beta\mu\nu} = {}^x F_{i\mu\nu} (X^i)^\alpha_{\beta} + {}^y F_{i\mu\nu} (Y^i)^\alpha_{\beta}, \quad (\text{IV.18})$$

where [18]

$${}^x F_{i\mu\nu} = (X_i)^\alpha_{\beta} R^\alpha_{\beta\mu\nu}. \quad (\text{IV.19})$$

If the final form of the curvature R as found above (IV.16) is substituted here, the Yang-Mills fields are ultimately found as

(IV.20)

$${}^x F_{i\mu\nu} = 2c^{-2} \{ (X_i)_{\mu\nu} p^2 + (X_i)_{\lambda\nu} p^\lambda p_\mu - (X_i)_{\lambda\mu} p^\lambda p_\nu \}.$$

The original assertion says now that the fields ${}^x F$ (IV.20) are just of the trivializable kind (II.10). In other words, there must exist some $\text{SO}(3)$ tensor fields ${}^x B_{i\mu}$ (II.3) such that (i) the field strengths (IV.20) are composed by them according to

$${}^x F_{i\mu\nu} = -\varepsilon_i^{jk} {}^x B_{j\mu} {}^x B_{k\nu}, \quad (\text{IV.21})$$

and (ii) the automorphism $[B]$ (IV.3) can be expressed by ${}^x B_{i\mu}$ with regard of (IV.15) as

$$B_{\mu\nu} = c^{-2} \hat{P}_{\mu\nu} = g^{ij} {}^x B_{i\mu} {}^x B_{j\nu}. \quad (\text{IV.22})$$

Indeed, the (unique) solution of the problem (IV.21, 22) for ${}^x B_{i\mu}$ in terms of p is easily found as

$${}^x B_{i\mu} = 2c^{-1} (X_i)_{\nu\mu} p^\nu. \quad (\text{IV.23})$$

Thus, the conformal solutions of the Yang-Mills equations really turn out as the $\mathfrak{SU}(2)$ part of a certain Riemannian conformal structure over E_4 . The goal here is that we have found, by the way, the tensor object ${}^x B_{i\mu}$ (IV.23) which is associated to the (trivializable) potential ${}^x A_{i\mu}$ (III.26a). With this pair of fields $\{{}^x A_{i\mu}, {}^x B_{i\mu}\}$ at hand, it is now an easy matter to determine the most general solution of the conformal type.

4. Di-Meron Solution

As was already pointed out in Sect. II, the necessary (and sufficient) conditions upon the pair of fields $\{{}^x A_{i\mu}, {}^x B_{i\mu}\}$ for representing a trivializable solution to the free Yang-Mills equations are (i) the special shape (II.10) of the field strengths $F_{i\mu\nu}$ and (ii) the symmetry relation (II.11) of the covariant derivative of the tensor objects $B_{i\mu}$.

Therefore, we introduce the potential ${}^x A_{i\mu}$ (III.26a) and the tensor objects ${}^x B_{i\mu}$ (IV.23) into (II.10) and (II.11) and then obtain, by use of some auxiliary relations such as

$$\begin{aligned} & -2\varepsilon_i^{jk} (X_j)_{\nu\mu} (X_k)_{\varrho\sigma} \\ & = (X_i)_{\nu\sigma} g_{\varrho\mu} + (X_i)_{\varrho\nu} g_{\mu\sigma} - (X_i)_{\mu\sigma} g_{\varrho\nu} - (X_i)_{\varrho\mu} g_{\nu\sigma} \end{aligned}$$

a pair of coupled non-linear, first-order equations for the vector fields $q(x)$, $p(x)$. These equations may be decoupled by putting

$$\begin{aligned} u &= \frac{1}{2} (c^{-1} p + q), \quad (\text{a}) \\ v &= \frac{1}{2} (q - c^{-1} p), \quad (\text{b}) \end{aligned} \quad (\text{IV.24})$$

where each of the new vector fields $u(x)$, $v(x)$ satisfies the same equation, namely

$$\partial_\mu w^\nu = 2w^\nu w_\mu - g_\mu^\nu (w^\lambda w_\lambda). \quad (\text{IV.25})$$

By contracting this equation with w^μ , one is easily convinced that the integral curves of the solution $w(x)$ are straight lines (with respect to the canonical connection $\hat{\omega}$ over E_4) and hence the only topolog-

ically non-trivial solution is a hedgehog field

$$\mathbf{w}(x) = \frac{\mathbf{x} - \mathbf{a}}{|\mathbf{x} - \mathbf{a}|^2}. \quad (\text{IV.26})$$

Since the center (\mathbf{a}) of the hedgehog field \mathbf{w} is arbitrary due to the translational invariance of (IV.25), we can have two different centres for the field $\mathbf{u}(x)$ and $\mathbf{v}(x)$ such that the original fields

$$\mathbf{q} = \frac{\mathbf{x} - \mathbf{a}}{|\mathbf{x} - \mathbf{a}|^2} + \frac{\mathbf{x} - \mathbf{b}}{|\mathbf{x} - \mathbf{b}|^2}, \quad (\text{a}) \quad (\text{IV.27})$$

$$c^{-1} \mathbf{p} = \frac{\mathbf{x} - \mathbf{a}}{|\mathbf{x} - \mathbf{a}|^2} - \frac{\mathbf{x} - \mathbf{b}}{|\mathbf{x} - \mathbf{b}|^2} \quad (\text{b})$$

represent the most general, trivializable solution to the present problem. In other words: The potentials ${}^x A_{i\mu}$, supplied by the vector field $\mathbf{q}(x)$ (IV.27 a), represent the most general, trivializable solution to the free Yang-Mills equations of the conformal type. The associate tensor objects ${}^x B_{i\mu}$ are then given by (IV.23) in connection with the vector field $\mathbf{p}(x)$ (IV.27b). The latter vector field is covariantly constant (cf. (IV.5)) with respect to the conformally flat connection Γ (III.13) and plays an important part for the geometric representation of the present solution. Finally, one directly checks that the conformal factor χ^2 really agrees with the square (p^2) of the characteristic vector \mathbf{p} as required by the normalization condition (IV.4). To this end, one first contracts the (ordinary) derivative of \mathbf{p}

$$\partial_\mu p_\nu = q_\mu p_\nu + q_\nu p_\mu - g_{\mu\nu} (p^\lambda q_\lambda) \quad (\text{IV.28})$$

with p^ν and finds

$$\frac{p_{,\nu}}{p} = q_\nu, \quad (\text{IV.29})$$

which compares to (III.13b). Next, one verifies for q_ν (IV.27a)

$$q_\nu = \partial_\nu \ln \chi, \quad (\text{a})$$

$$\chi = \frac{c^2}{|\mathbf{x} - \mathbf{a}| \cdot |\mathbf{x} - \mathbf{b}|}. \quad (\text{b}) \quad (\text{IV.30})$$

On the other hand, the Euclidean length of \mathbf{p} (IV.27b) is

$$p = c \left(\frac{|\mathbf{a} - \mathbf{b}|}{|\mathbf{x} - \mathbf{a}| \cdot |\mathbf{x} - \mathbf{b}|} \right). \quad (\text{IV.31})$$

Hence, we really can have $\chi \equiv p$ if we choose

$$c = |\mathbf{a} - \mathbf{b}|. \quad (\text{IV.32})$$

V. Geometrical Representation

The general conformal solution obtained above exhibits some nice geometric properties which we want to mention briefly.

1. Projective Properties

First, remember that the automorphism $[B]$ has turned out as the identity \hat{P} (IV.15) in $\hat{\mathcal{A}}$. On the other hand, the general shape of the (gauge independent) fields $B_{\mu\nu}$ (IV.3a) may be found by use of the extrinsic curvature fields $B_{i\mu}$ in the positive gauge (II.20). Hence, they read in terms of the unit normal $\hat{\mathbf{n}}$ (II.19)

$$B_{\mu\nu} = (\partial_\mu \hat{\mathbf{n}}^x) (\partial_\nu \hat{\mathbf{n}}_x). \quad (\text{V.1})$$

Therefore, the right hand side of (V.1) is proportional to the (Euclidean) projector $P_{\mu\nu} := \chi^{-2} \hat{P}_{\mu\nu}$

$$(\partial_\mu \hat{\mathbf{n}}^x) (\partial_\nu \hat{\mathbf{n}}_x) = \left(\frac{\chi}{c} \right)^2 P_{\mu\nu}. \quad (\text{V.2})$$

Using the projector property $P \cdot P = P$, the alternative combination of the derivatives of \mathbf{n} is identical to the projector (\bar{P}) onto the distribution $\bar{\mathcal{A}}$ introduced at the end of Sect. II:

$$(\partial_\mu \hat{\mathbf{n}}^x) (\partial^\mu \hat{\mathbf{n}}_\beta) = \left(\frac{p}{c} \right)^2 \bar{P}^x_\beta, \quad (\text{a})$$

$$\bar{P}^x_\beta \bar{P}^\beta_\gamma = \bar{P}^x_\gamma, \quad (\text{b})$$

$$\bar{P}^x_\beta \hat{\mathbf{n}}^\beta = 0. \quad (\text{c}) \quad (\text{V.3})$$

In order to obtain from here the final equation for the normal field $\hat{\mathbf{n}}(x)$, one has to eliminate the conformal factor p^2 by contraction of indices α, β

$$(\partial_\nu \hat{\mathbf{n}}^\gamma) (\partial^\nu \hat{\mathbf{n}}_\gamma) = -(\hat{\mathbf{n}}^\gamma \square \hat{\mathbf{n}}_\gamma) = 3 \left(\frac{p}{c} \right)^2. \quad (\text{V.4})$$

One then ultimately finds

$$(\partial_\mu \hat{\mathbf{n}}^x) (\partial^\mu \hat{\mathbf{n}}_\beta) + \frac{1}{3} (\hat{\mathbf{n}}^\gamma \square \hat{\mathbf{n}}_\gamma) (g^x_\beta + \hat{\mathbf{n}}^x \hat{\mathbf{n}}_\beta) = 0. \quad (\text{V.5})$$

This equation, supplied by the appropriate boundary conditions upon the normal $\hat{\mathbf{n}}$ to $\bar{\mathcal{A}}$, uniquely determines that distribution $\bar{\mathcal{A}}$. Although (V.5) is merely a first-order equation, we shall prefer to compute the unit normal $\hat{\mathbf{n}}$ in a somewhat simpler way by use of a quaternion representation. We return to the first-order problem (V.5) when discussing the multi-meron case in the next section.

2. Quaternion Formalism

We have mentioned in Sect. II that there exists a gauge ${}^+A_{i\mu}$ (“positive gauge”) for which the trivializable solutions are immediately expressed in terms of the unit normal \hat{n} through (II.18, 20). A remarkable property of the positive gauge consists in the fact that it is half the value of a trivial connection (cf. (II.22)). We combine now these two features of the positive gauge by rewriting the generating SU⁺(2) element X (II.16) in terms of the unit normal \hat{n} by using the real, four-dimensional quaternion representation for the groups SU[±](2). Introducing the quaternion basis for SU⁺(2) by Ξ^μ ,

$$\{\Xi^\mu\} \equiv \{\Xi^0; \Xi^i\} = \{1; 2X^i\}, \quad (\text{V.6})$$

and, analogously, the anti-quaternion basis for SU[−](2) by H^μ ,

$$\{H^\mu\} \equiv \{H^0; H^i\} = \{1; 2Y^i\}, \quad (\text{V.7})$$

we find the quaternion composition relations, such as

$$\Xi^\mu \cdot \Xi^\nu = (H^\nu)^{\mu}_{\lambda} \Xi^\lambda, \quad \text{etc.} \quad (\text{V.8})$$

The quaternions are related to the anti-quaternions through “space inversion” Σ :

$$\Xi^\mu = \Sigma^\mu_{\nu} \Xi^\nu = \Sigma \cdot H^\mu \cdot \Sigma, \quad (\text{a})$$

$$\Sigma^\mu_{\nu} = \text{diag}[1, -1, -1, -1]. \quad (\text{b}) \quad (\text{V.9})$$

Further, we introduce the conjugate basis through

$$\{\Xi^\mu\} = \{1; -2X^i\}, \quad (\text{a})$$

$$\{\hat{H}^\mu\} = \{1; -2Y^i\}, \quad (\text{b}) \quad (\text{V.10})$$

and then find commutation relations of the following kind:

$$\Xi^\mu \cdot \Xi^\nu + \Xi^\nu \cdot \Xi^\mu = -2g^{\mu\nu} \cdot 1, \quad (\text{a})$$

$$\Xi^\mu \cdot \Xi^\nu - \Xi^\nu \cdot \Xi^\mu = 8(Y_j)^{\mu\nu} X^j, \quad (\text{b})$$

$$\Xi^\mu \cdot \hat{\Xi}^\nu - \hat{\Xi}^\nu \cdot \Xi^\mu = 8(X_j)^{\mu\nu} X^j. \quad (\text{c}) \quad (\text{V.11})$$

With the aid of this quaternion formalism it becomes a simple matter to compute the normal $\hat{n}(x)$ for the previous di-meron solution. One simply writes the generating SU⁺(2) element X (II.16) in quaternion form as

$$X = -\hat{n}_\mu \Xi^\mu. \quad (\text{V.12})$$

Once the generating matrix X is known by putting the trivializable solution into the positive gauge

(II.22), the unit normal \hat{n} is readily deduced from X through

$$\hat{n}^\mu = \frac{1}{4} \text{tr}(\Xi^\mu \cdot X). \quad (\text{V.13})$$

3. Di-Meron Surfaces

Applying this procedure to the present di-meron solution, one starts with the “x-gauge” for $A_{i\mu}$ (III.26a) and $B_{i\mu}$ (IV.23):

$${}^x\mathcal{A}_\mu = \frac{1}{2} (X_a^{-1} \partial_\mu X_a + X_b^{-1} \partial_\mu X_b), \quad (\text{a})$$

$${}^x\mathcal{B}_{i\mu} X^i \equiv {}^x\mathcal{B}_\mu = \frac{1}{2} (X_a^{-1} \partial_\mu X_a - X_b^{-1} \partial_\mu X_b). \quad (\text{b}) \quad (\text{V.14})$$

The matrices X_a, X_b are given by

$$X_a = -\frac{x_\mu - a_\mu}{|\mathbf{x} - \mathbf{a}|} \Xi^\mu \equiv -\hat{u}_\mu \Xi^\mu, \quad (\text{a})$$

$$X_b^{-1} = -\frac{x_\mu - b_\mu}{|\mathbf{x} - \mathbf{b}|} \Xi^\mu \equiv -\hat{v}_\mu \Xi^\mu, \quad (\text{b}) \quad (\text{V.15})$$

$$(\hat{u}^\mu \hat{u}_\mu = \hat{v}^\mu \hat{v}_\mu = -1).$$

These matrices describe single meron solutions centered around $\mathbf{x} = \mathbf{a}$ and $\mathbf{x} = \mathbf{b}$, resp., where X^{-1} generates the corresponding anti-meron configuration. The unit normal \hat{n} for the di-meron solution is found now by passing over to the positive gauge, which is accomplished by applying the gauge transformation X_b^{-1} to (V.14). This readily yields the positive gauge (II.20) as desired and the generating matrix becomes $X_{a,b} = X_a \cdot X_b^{-1}$. Thus, using the quaternion decomposition rules such as (V.8), the normal \hat{n} (V.13) is readily found as

$$\begin{aligned} \hat{n}_\lambda &= -(\Xi_\lambda)^{\mu\nu} \frac{x_\mu - a_\mu}{|\mathbf{x} - \mathbf{a}|} \cdot \frac{x_\nu - b_\nu}{|\mathbf{x} - \mathbf{b}|} \\ &\equiv -(\Xi_\lambda)^{\mu\nu} \hat{u}_\mu \hat{v}_\nu. \end{aligned} \quad (\text{V.16})$$

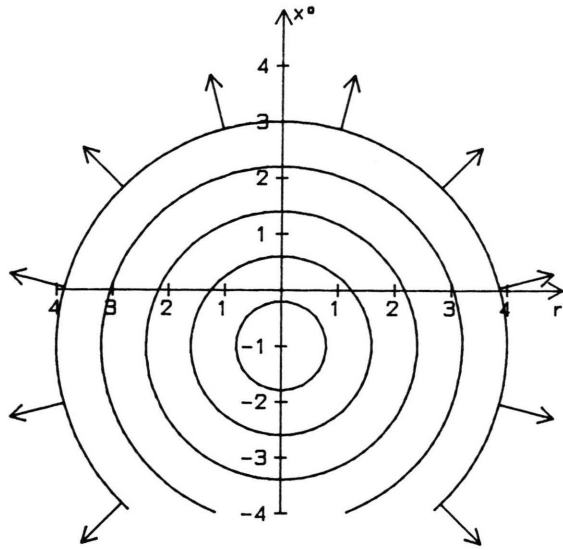
If one of the two singular points is removed to infinity ($|\mathbf{b}| \rightarrow \infty$, say), we are left with a single meron configuration [20]: $X_{a,b} \rightarrow C \cdot X_a$, whose normal \hat{n}_+ is

$$\hat{n}_+ = \hat{u} = \frac{\mathbf{x} - \mathbf{a}}{|\mathbf{x} - \mathbf{a}|}. \quad (\text{V.17})$$

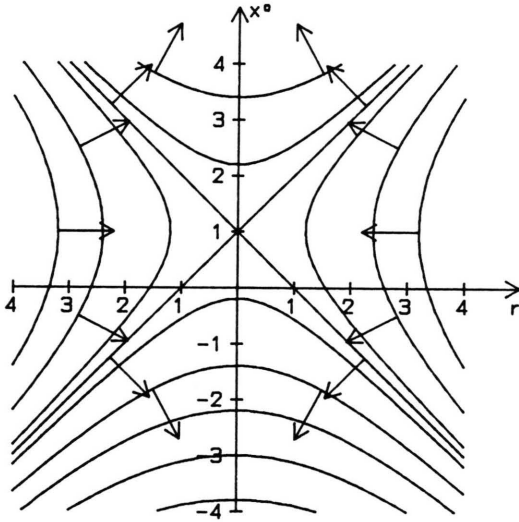
Hence the integral surfaces of $\bar{\mathcal{A}}$ become 3-spheres centered around \mathbf{a} (Fig. 1a):

$$g_{\mu\nu}(x^\mu - a^\mu)(x^\nu - a^\nu) = \text{const.} \quad (\text{V.18})$$

Conversely, removing the meron at \mathbf{a} to infinity ($|\mathbf{a}| \rightarrow \infty$), we are left with an anti-meron configura-



a)



b)

Fig. 1.

tion at \mathbf{b} : $X_{a,b} \rightarrow C \cdot X_b^{-1}$, whose normal $\hat{\mathbf{n}}_-$ is obtained through space inflection of the hedgehog field $\hat{\mathbf{v}}$:

$$\hat{\mathbf{n}}_-^\lambda = \Sigma_\mu^\lambda \hat{v}^\mu. \quad (\text{V.19})$$

Here, the space inflection Σ is due to the transition to the conjugate basis (V.10), which is necessary in order to reproduce the standard form (V.12) for X . Therefore the anti-meron configuration is formed by a set of 3-hyperbolas centered around \mathbf{b}

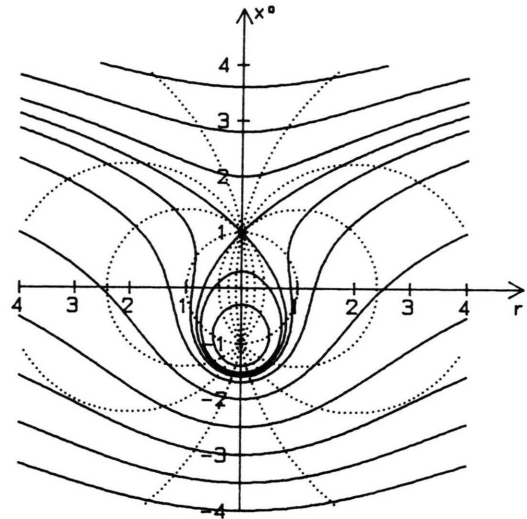


Fig. 2. Integral surfaces for the single meron cases (Fig. 1, (V.18, 19)) are melted into the di-meron surfaces (V.21) (solid lines). The di-meron surfaces are asymptotically flat. The dotted lines represent the meron bags consisting of the characteristic lines.

(Fig. 1 b):

$$\Sigma_{\mu\nu}(x^\mu - b^\mu)(x^\nu - b^\nu) = \text{const.} \quad (\text{V.20})$$

Observe that the meron and anti-meron cases (V.18, 20) are related to each other by simply interchanging the Euclidean and Minkowski metric tensors $g \leftrightarrow \Sigma$. Clearly, the use of Σ (V.9 b) singles out the “time” direction as the SO(3) symmetry axis.

The SO(3) symmetry around the straight line connecting the two meron centres at \mathbf{a} and \mathbf{b} , resp., may now be exploited by looking for the analytic form of the di-meron surfaces. The di-meron normal $\hat{\mathbf{n}}$ (V.16) is composed in a simple way by the corresponding single-meron normals $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$. Consequently, the di-meron surfaces are obtained after some elementary, differential geometric computations as (see Fig. 2)

$$g_{\mu\nu}(x^\mu - a^\mu)(x^\nu - a^\nu) = \text{const} \cdot \exp \left[2 \frac{(a^\mu - b^\mu)x_\mu}{|\mathbf{a} - \mathbf{b}|^2} \right]. \quad (\text{V.21})$$

The present geometric picture admits also a concrete interpretation of those “meron bags” introduced by Glimm and Jaffe [12]. According to their definition, these bags are revealed in our geometric model as those 3-surfaces which have a vanishing

“time” component of the normal \hat{n} : $\hat{n}^0(x) = \cos v/2 = 0$ (see Fig. 2). Thus, the meron bags turn out as special cases of a more general set of 3-surfaces, which are built by the characteristic lines. Remember the fact that the unit normal \hat{n} is parallel transported along any characteristic line and hence the $(\hat{n}^0(x) = 0)$ -bag is a special case within the larger set of bags given by $\hat{n}^0(x) = \text{const}$. For the di-meron solution, the bags determined by $\hat{n}^0(x) = \text{const}$ are all those 3-spheres containing the two meron centers (Figure 2). Moreover, in the special case of the di-meron solution, the characteristic lines have been turned out as geodesics of the conformally flat metric $G_{\mu\nu}$. This, however, leads to a very peculiar situation: The di-meron bag emerges now as a closed surface built by characteristic lines along which both the unit normal \hat{n} and the characteristic vector \mathbf{p} are parallel transported! (The parallel transport of \hat{n} refers to the canonical connection over E_4 , whereas the parallel transport of \mathbf{p} is referred to the conformally flat connection $\tilde{\Gamma}$.) It would be interesting to see whether this property survives the generalization to a generic multi-meron solution containing more than two merons.

VI. Multi-Meron Solutions

Despite the fact that the main interest of the present paper was concentrated on the di-meron solutions, we want to briefly look also at the multiple-meron configurations. Here, we restrict ourselves to the case where the centers of all the merons are placed along a straight line (x^0 -axis, say). It is well-known that the expected solutions really do exist [21]. Although the analytic form of these general solutions could not be discovered within the present geometric framework, it is very instructive to see how the previous di-meron solution emerges as a special case of the more general multiple meron configuration.

Of course, the relevant geometric objects such as extrinsic curvature B , Riemannian R and characteristic vector \mathbf{p} can no longer be covariantly constant here with respect to the conformally flat connection $\tilde{\Gamma}$. Observe that we have shown above that the di-meron solution is the only one compatible with the (conformally) covariant constancy condition. However, we shall readily show that it suffices to give up this constancy condition (cf. (III.8)) whereas

the more important properties of trivializability and of conformal flatness of the Riemannian structure persist in the generic multiple meron situation.

In case of the SO(3) symmetric arrangement of merons, we again try our trivializable ansatz and observe that the angle v (II.19) depends here solely upon the “spatial” distance r ($\equiv \sqrt{-g_{ij}x^ix^j}$) and “time” t ($\equiv x^0$): $v = v(t, r)$. Further, the SO(3) gauge scalarfield $\tilde{v}^i(x)$ coincides with the “space-like” unit position vector \tilde{x}^i : $\tilde{v}^i(x) \equiv \tilde{x}^i$ ($\tilde{x}^i\tilde{x}_i = -1$). The external SO(3) symmetry, inherent in such a configuration, is expressed geometrically by the fact that the “space-like” 2-distribution \tilde{A} for which the gauge scalar $\tilde{v}^i(x)$ represents the unit normal, is umbilical [22] with respect to \tilde{v}^i :

$$D_\mu \tilde{v}^i = \frac{\cos \frac{v}{2}}{r} \tilde{P}^i_\mu. \quad (\text{VI.1})$$

Here, \tilde{P} denotes the projector onto \tilde{A} :

$$\tilde{P}_{\mu\nu} = g_\mu^i g_\nu^j \tilde{P}_{ij}, \quad (\text{a})$$

$$\tilde{P}_{ij} = g_{ij} + \tilde{v}_i \tilde{v}_j. \quad (\text{b}) \quad (\text{VI.2})$$

By virtue of the simple result (VI.1) for the covariant derivative of \tilde{v}^i the extrinsic curvature $B_{i\mu}$ (II.30) is essentially determined by the angle $v(t, r)$. Hence, its generally covariant derivative may be computed and is found after some lengthy calculations as

$$\begin{aligned} \mathcal{D}_\sigma B_{i\mu} &= D_\sigma B_{i\mu} - B_{i\lambda} \Gamma^\lambda_{\mu\sigma} \\ &= M_{\mu\sigma} \tilde{v}_i, \end{aligned} \quad (\text{VI.3})$$

where

$$\begin{aligned} M_{\mu\sigma} &= \hat{t}_\mu \hat{t}_\sigma \left(\ddot{\kappa} - \frac{\kappa'}{r} + \frac{\kappa'^2 - \dot{\kappa}^2}{\tan \kappa} \right) \\ &\quad + (\hat{t}_\mu \hat{r}_\sigma + \hat{t}_\sigma \hat{r}_\mu) \left(\dot{\kappa}' + \frac{\dot{\kappa}}{r} - 2 \frac{\dot{\kappa} \kappa'}{\tan \kappa} \right) \\ &\quad + \hat{r}_\mu \hat{r}_\sigma \left(\kappa'' + \frac{\kappa'}{r} + \frac{\dot{\kappa}^2 - \kappa'^2}{\tan \kappa} \right) \\ &\quad + \cot \kappa \tilde{P}_{\mu\sigma} \left(\frac{\sin^2 \kappa}{r^2} - \kappa'^2 - \dot{\kappa}^2 \right). \end{aligned} \quad (\text{VI.4})$$

Here, we have used $\kappa \equiv \frac{v}{2}$, $\dot{\kappa} := \frac{\partial \kappa}{\partial t}$, $\kappa' := \frac{\partial \kappa}{\partial r}$.

The unit vectors with respect to “time” t and radial distance r have been denoted by \hat{t}_μ and \hat{r}_μ , resp.

($\hat{t}^\mu \hat{t}_\mu = \hat{r}^\mu \hat{r}_\mu = -1$, $\hat{t}^\mu \hat{r}_\mu = 0$). The important point of the result (VI.3) consists in the fact that the connection coefficients $F_{\mu\sigma}^\lambda$ emerging there, are still due to a conformally flat metric G where the conformal factor χ turns out as (cf. (III.11, 12))

$$\chi = \frac{\sin \kappa}{r}. \quad (\text{VI.5})$$

In order to satisfy now the Yang-Mills equations (III.1), we form the combination of extrinsic curvature derivatives needed there and find by use of (VI.3)

$$D_\mu B_{j\nu} - g_{\mu\nu} (D^\lambda B_{j\lambda}) = B_{j\lambda} T_{\nu\mu}^\lambda + \bar{v}_j N_{\nu\mu}, \quad (\text{VI.6})$$

where

$$N_{\nu\mu} = M_{\nu\mu} - g_{\nu\mu} M^\lambda_{\lambda}. \quad (\text{VI.7})$$

However, as discussed below (III.1), the first term containing the totally symmetric tensor $T_{\lambda\nu\mu}$ satisfies the field equations automatically and we are left with the second term proportional to \bar{v}_j . Therefore, the extrinsic curvature B_k^μ in front of the bracket term of the field equations (III.1) may be replaced by its covariant derivative part ($D^\mu \bar{v}_k$), and the field equations finally read

$$D^\mu F_{i\mu\nu} = -\tan \kappa \varepsilon_i^{kj} \bar{v}_j (D^\mu \bar{v}_k) N_{\nu\mu} = 0. \quad (\text{VI.8})$$

Because of the umbilicality property (VI.1) we have

$$\tilde{P}_\lambda^\mu D^\lambda \bar{v}_k = D^\mu \bar{v}_k, \quad (\text{VI.9})$$

which means that only the projection $\tilde{N}_{\nu\mu}$ of the tensor $N_{\nu\mu}$ onto $\tilde{\Delta}$ becomes active in the field equations (VI.8). Hence, it must vanish:

$$0 = \tilde{N}_{\nu\mu} := \tilde{P}_\nu^\sigma N_{\sigma\varrho} \tilde{P}_\mu^\varrho. \quad (\text{VI.10})$$

After a short computation, one finds by use of (VI.4)

$$\tilde{N}_{\nu\mu} = \tilde{P}_{\nu\mu} \left(\ddot{\kappa} + \kappa'' + \frac{\dot{\kappa}^2 + \kappa'^2}{\tan \kappa} - \frac{\sin \kappa \cos \kappa}{r^2} \right), \quad (\text{VI.11})$$

and thus the Yang-Mills equations for our trivializable ansatz reduce to

$$\ddot{\kappa} + \kappa'' + \frac{\dot{\kappa}^2 + \kappa'^2}{\tan \kappa} - \frac{\sin \kappa \cos \kappa}{r^2} = 0. \quad (\text{VI.12})$$

Finally, putting here $\psi := \cos \kappa$, the Yang-Mills equations (IV.12) assume the shape

$$\ddot{\psi} + \psi'' + r^{-2} \psi (1 - \psi^2) = 0, \quad (\text{VI.13})$$

which just agrees with the multiple meron equation of Glimm and Jaffe [12]. So we see that the SO(3)

symmetric multiple meron solutions really exhibit the property of trivializability. At present, it seems unclear to us whether this feature survives the generalization to a situation where the merons are dislocated arbitrarily over E_4 .

Now we return to the special case of the di-meron solution, for which the covariant constancy condition (III.8) demands that the tensor M vanishes in (VI.3). Since this tensor is composed of four linearly independent parts (VI.4), the di-meron solution must satisfy the following four non-linear partial differential equations:

$$\ddot{\kappa} - \frac{\kappa'}{r} + \frac{\kappa'^2 - \dot{\kappa}^2}{\tan \kappa} = 0, \quad (\text{a})$$

$$\kappa'' + \frac{\kappa'}{r} + \frac{\dot{\kappa}^2 - \kappa'^2}{\tan \kappa} = 0, \quad (\text{b})$$

$$\dot{\kappa}' + \frac{\dot{\kappa}}{r} - 2 \frac{\dot{\kappa} \kappa'}{\tan \kappa} = 0, \quad (\text{c})$$

$$\frac{\sin^2 \kappa}{r^2} - \kappa'^2 - \dot{\kappa}^2 = 0. \quad (\text{d}) \quad (\text{VI.14})$$

It is easily seen how this set of four equations leads to the single field equation (VI.12): Adding together the first two equations (VI.14a, b) yields the 2-dimensional Laplace equation

$$\ddot{\kappa} + \kappa'' = 0, \quad (\text{VI.15})$$

and if this is multiplied by $\tan \kappa$ and subtracted from the last equation (VI.14d) we just recover the field equation (VI.12). So we see that the di-meron solution is a highly special case which even satisfies the linear Laplace equation despite the fact that the Yang-Mills equations are intrinsically non-linear. Observe also that the last equation (d) of the set (VI.14) is of first order and hence compares to the first order equation (V.5) which holds only for the di-meron solution. Indeed, if the Euclidean equation (V.5), is written down for the present SO(3) symmetric case in terms of the angle $\kappa(t, r)$ it becomes identical to (VI.14d).

Finally, let us write down the solution to the system of equations (VI.14) in terms of the angle κ :

$$\begin{aligned} \kappa(t, r) &= \kappa^+(t, r) + \kappa^-(t, r), \\ \kappa^\pm(t, r) &= \pm \tan^{-1} \frac{r}{t^\pm}, \\ t^+ &= t + a, \\ t^- &= t - a. \end{aligned} \quad (\text{VI.16})$$

Clearly, this is the superposition of two merons located on the time axis at $t = \pm a$ (see Figure 2). Unfortunately the generic multiple meron solution

(more than two merons) is not such a simple superposition of the individual meron angles \varkappa (contrary to what has been maintained in literature [5c]).

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